Smaller Alignment Index (SALI): Determining the ordered or chaotic nature of orbits in conservative dynamical systems

Ch. Skokos

Department of Mathematics,
University of Patras,
Patras, Greece, and

Research Center for Astronomy,
Academy of Athens,
Athens, Greece

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Outline

- **Methods for distinguishing between ordered and chaotic motion**

- **Definition of the smaller alignment index (SALI)**

- **Applications**
  - **Symplectic maps**
    - **2D maps**
    - **4D maps**
    - **A case of weak chaos in a 4D map**
  - **Hamiltonian flows**
    - **Hénon Heiles system (2D Hamiltonian)**

- **Conclusions**

Work in collaboration with:
Ch. Antonopoulos, T. C. Bountis & M. N. Vrahatis
(Department of Mathematics, University of Patras)

Problem: Distinguishing between ordered and chaotic motion in dynamical systems.

- **Inspection of the consequents of an orbit on a Poincaré surface of section.**

- **The maximal Lyapunov Characteristic Number (LCN):** the limit of the finite time Lyapunov characteristic number:

$$L_t = \frac{1}{t} \ln \frac{|\xi_t|}{|\xi_0|}$$

- **The frequency analysis method**

- **The fast Lyapunov indicator (FLI)**

- **The study of spectra of dynamical quantities like the "short time Lyapunov characteristic numbers" or "stretching numbers"**

- **The power spectrum of geodesic divergences**

Definition of the alignment indices

Let us consider the **n-dimensional phase space of a conservative dynamical system**, which could be a **symplectic map** or a **Hamiltonian flow.**

We consider also an orbit in that space with initial condition

$$P(0)=(x_1(0), x_2(0),\ldots,x_n(0))$$

and a infinitesimal **deviation vector**

$$\mathbf{v(0)=(dx_1(0), dx_2(0),\ldots, dx_n(0))}$$

from the initial point P(0).

The evolution in time (in maps the time is discrete and is equal to the number N of the iterations) of two different initial deviation vectors (e.g. $v_1(0)$, $v_2(0)$), is defined by:

- the equations of the **tangent map** (for mappings) and
- the **variational equations** (for flows)

Following
Skokos Ch.: 2001, J. Phys. A, 34, 10029-10043
we define as **parallel alignment index**, the quantity:

$$\mathbf{d_- \equiv \|v_1(t) - v_2(t)\|}$$

and as **antiparallel alignment index**, the quantity:

$$\mathbf{d_+ \equiv \|v_1(t) + v_2(t)\|}$$

where $\| \cdot \|$ denotes the Euclidean norm of a vector.

The **smaller alignment index (SALI)** is

$$\mathbf{SALI = \min(d_-, d_+)}$$

We consider the vectors $v_1(t)$ and $v_2(t)$ to be normalized with norm equal to 1. Then the two vectors:

- **tend to coincide** when

$$\mathbf{d_- \to 0,\ d_+ \to 2,\ SALI \to 0}$$

- **tend to become opposite** when

$$\mathbf{d_- \to 2,\ d_+ \to 0,\ SALI \to 0}$$

Using the remark of Voglis et al. (1999, Cel. Mech. Dyn. Astr., 73, 211) that in

- 2D maps
  Any two deviation vectors tend to coincide or become opposite for ordered and chaotic orbits,
- 4D maps
  The ordered motion occurs on a 2D torus and two different initial deviation vectors become tangent to different directions on the torus.
  In chaotic cases two initially different deviation vectors tend to coincide to the direction defined by the most unstable nearby manifold,

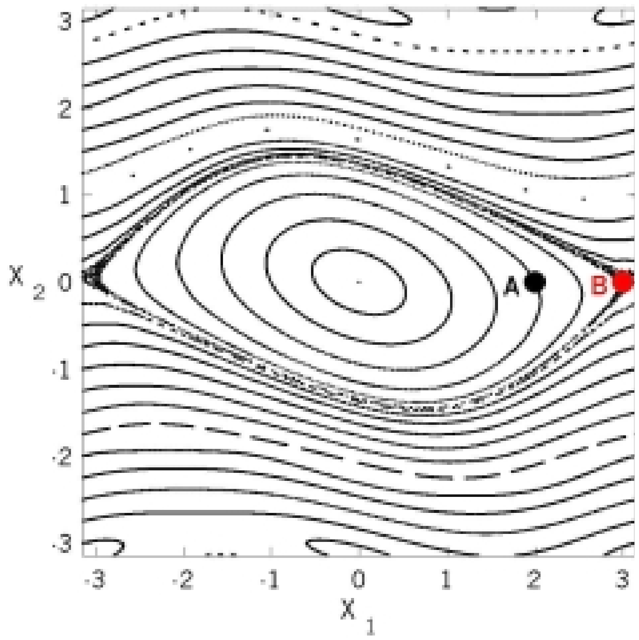
we conclude that for a system with an n-dimensional phase space we have the following cases:

n=2 (2D maps)
SALI→0 both in ordered and chaotic cases following completely different time rates which allows us to distinguish between the two cases.

n>2 (Hamiltonian flows and multidimensional maps)
SALI→0 for chaotic orbits, while
SALI→constant ≠ 0 for ordered orbits

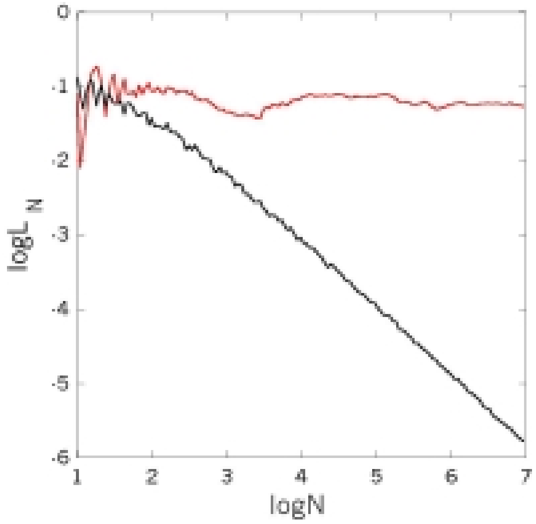
Ordered and chaotic orbits in a 2D map

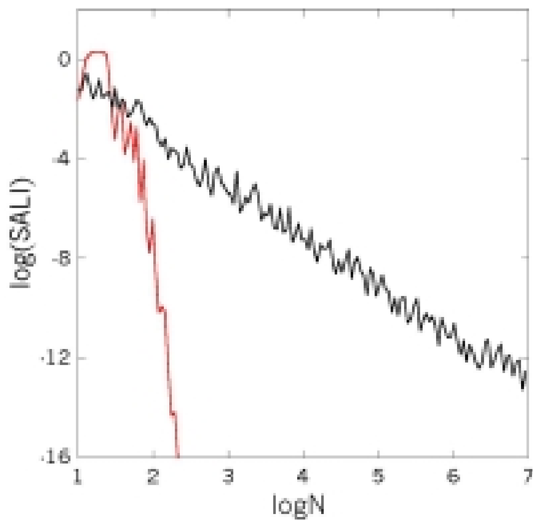
$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu \sin(x_1 + x_2) \end{aligned} \qquad (\text{mod } 2\pi)$$



For ν=0.5 we consider the orbits:

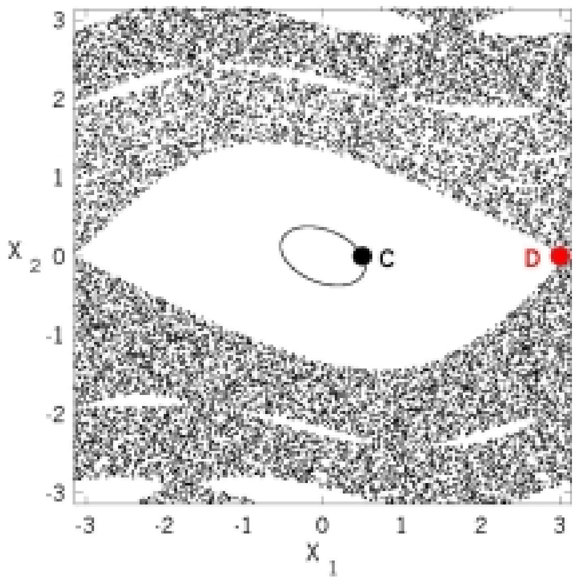
- ***ordered orbit A** with initial conditions $x_1=2$, $x_2=0$.*
- ***chaotic orbit B** with initial conditions $x_1=3$, $x_2=0$.*





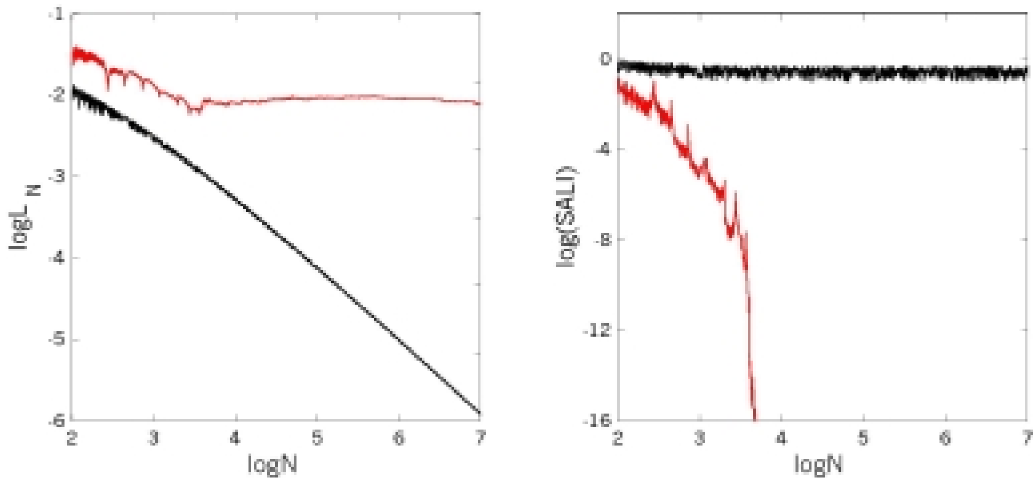
Ordered and chaotic orbits in a 4D map

$$
\begin{aligned}
x_1' &= x_1 + x_2 \\
x_2' &= x_2 - \nu \sin(x_1 + x_2) - \mu[1 - \cos(x_1 + x_2 + x_3 + x_4)] \\
x_3' &= x_3 + x_4 \\
x_4' &= x_4 - \kappa \sin(x_3 + x_4) - \mu[1 - \cos(x_1 + x_2 + x_3 + x_4)]
\end{aligned}
\qquad (\bmod 2\pi)
$$



For ν=0.5, κ=0.1, μ=0.1 we consider the orbits:

- ***ordered orbit C*** *with initial conditions* $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$.
- ***chaotic orbit D*** *with initial conditions* $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.

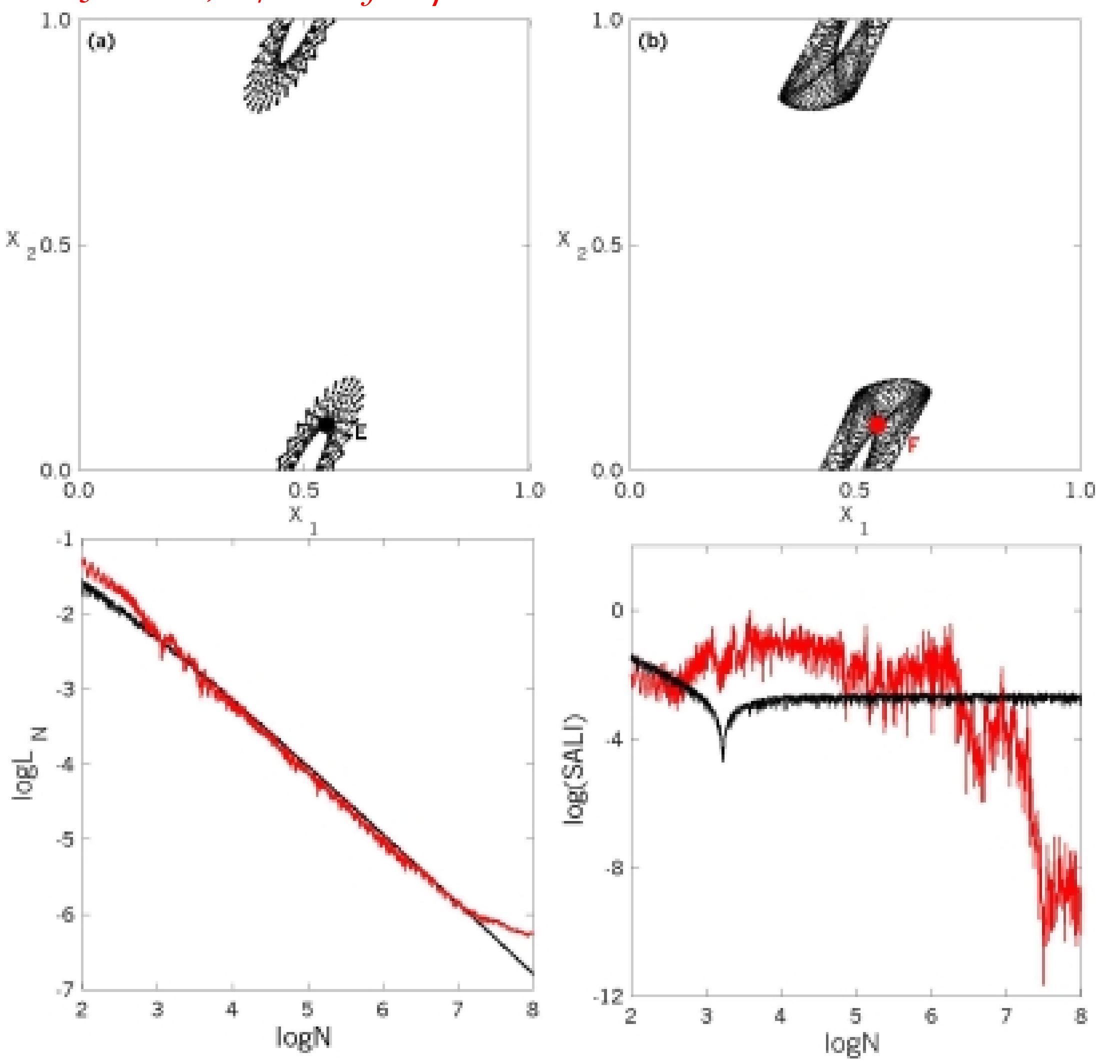


A case of weak chaos

$$\begin{aligned} x_1' &= x_1 + x_2' \\ x_2' &= x_2 + \frac{K}{2\pi}\sin(2\pi x_1) - \frac{\beta}{\pi}\sin[2\pi(x_3 - x_1)] \\ x_3' &= x_3 + x_4' \\ x_4' &= x_4 + \frac{K}{2\pi}\sin(2\pi x_3) - \frac{\beta}{\pi}\sin[2\pi(x_1 - x_3)] \end{aligned} \quad (\mathrm{mod}\, 1)$$

For K=3 we select two orbits with the same initial conditions

- ***ordered orbit E*** *with initial conditions $x_1=0.55$, $x_2=0.1$, $x_3=0.62$, $x_4=0.2$ for $\beta=0.1$.*
- ***chaotic orbit F*** *with initial conditions $x_1=0.55$, $x_2=0.1$, $x_3=0.62$, $x_4=0.2$ for $\beta=0.3051$.*

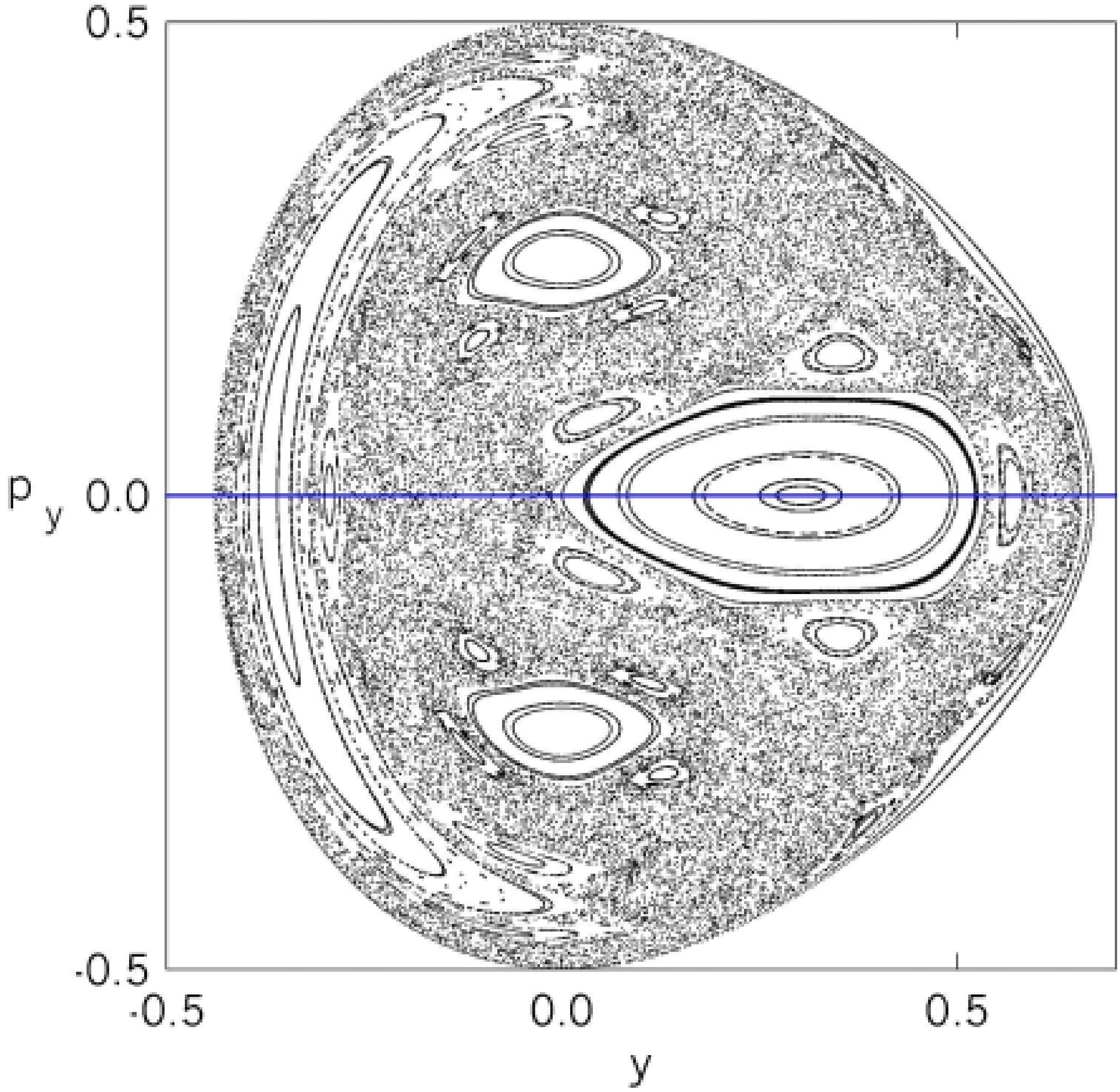


A Hamiltonian system of 2 degrees of freedom

Hénon Heiles system

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3} y^3$$

We consider the case for H = 1/8

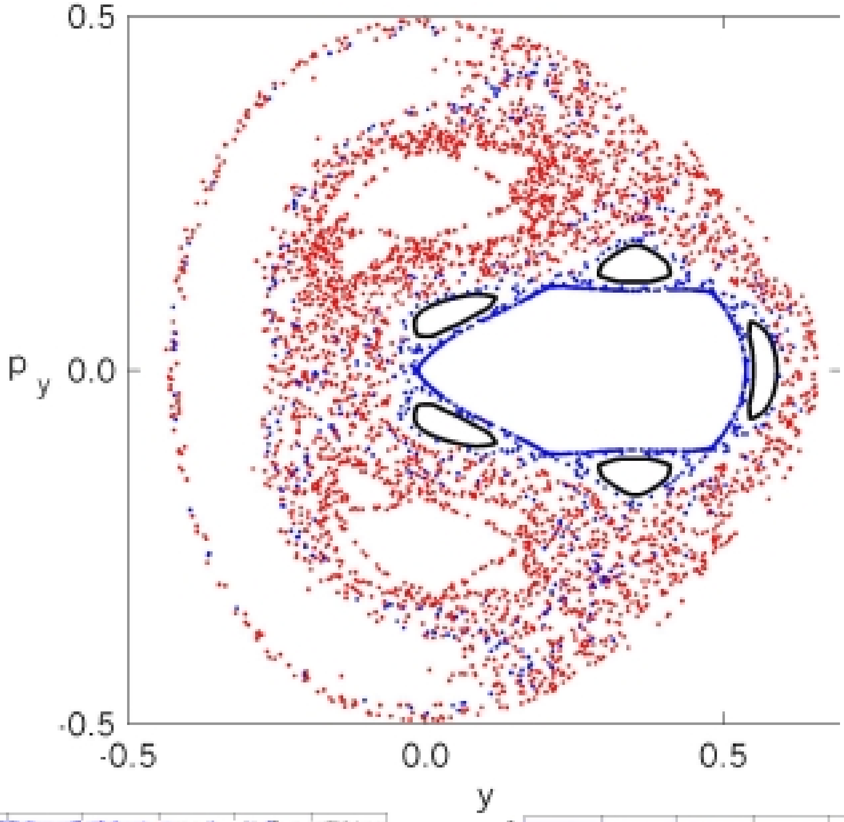


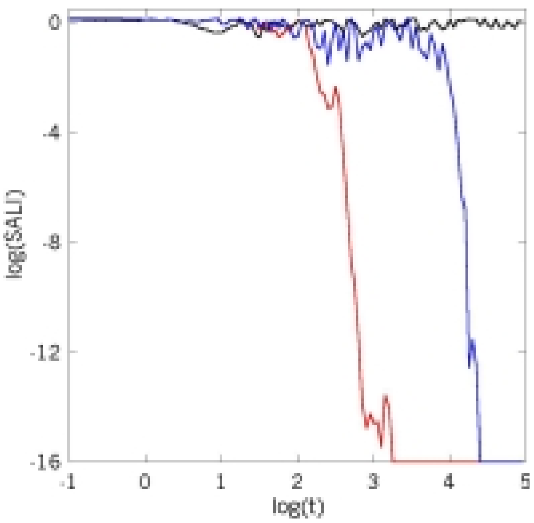
We consider the orbits with initial conditions:

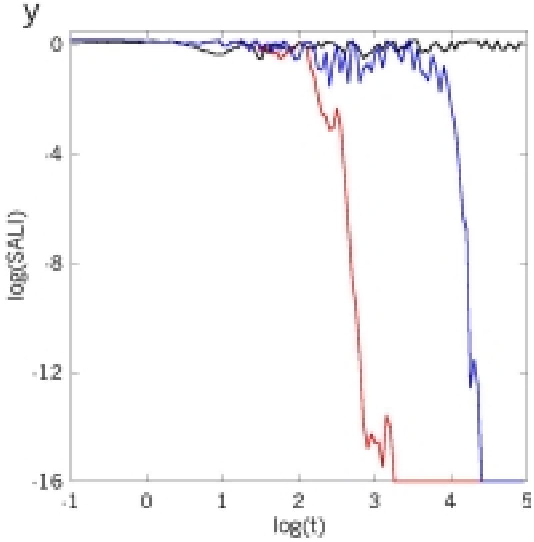
Ordered orbit, $x=0$, $y=0.55$, $p_x=0.2417$, $p_y=0$

Chaotic orbit, $x=0$, $y=-0.016$, $p_x=0.49974$, $p_y=0$

Chaotic orbit, $x=0$, $y=-0.01344$, $p_x=0.49982$, $p_y=0$

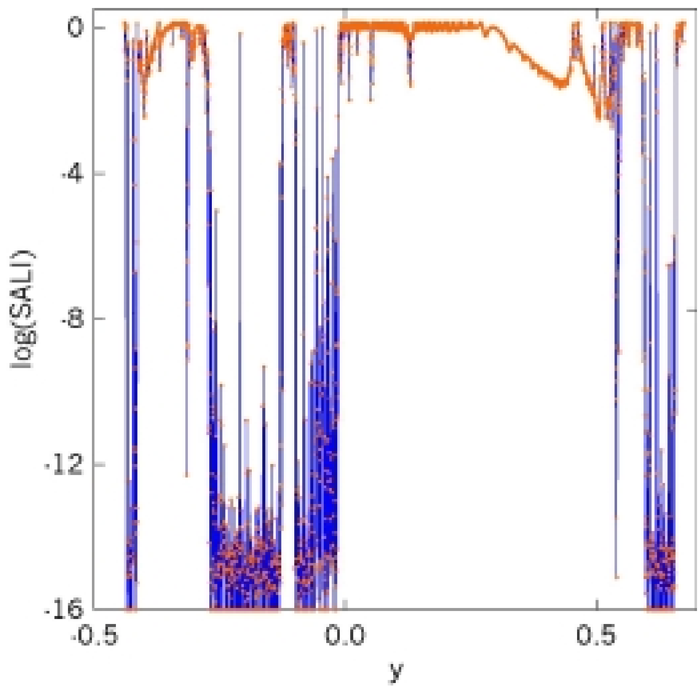




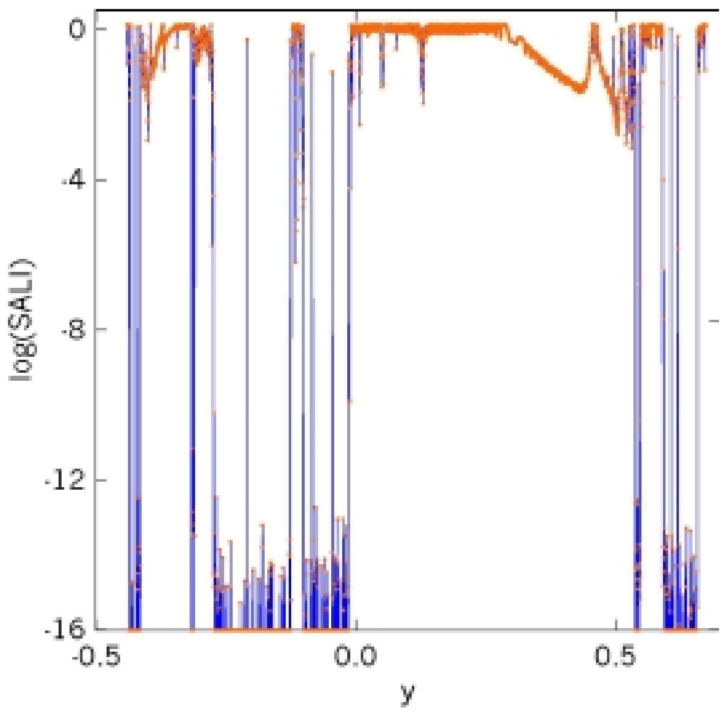


We compute the SALI for 5,000 orbits with initial conditions on the $p_y=0$ axis for:

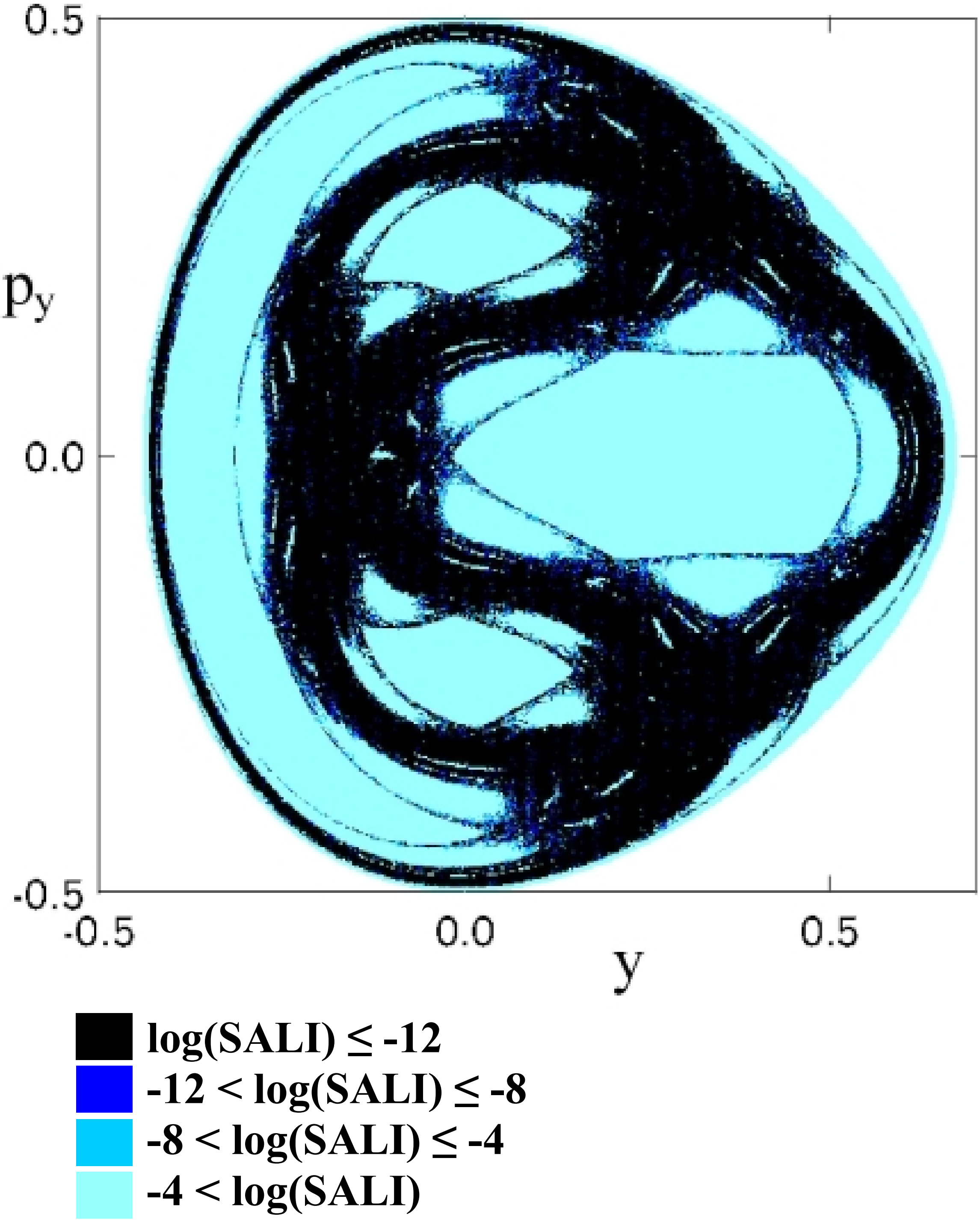
t=1,000



t=4,000



We compute the SALI for 562,500 orbits with initial conditions on the Poincaré surface of section (y, p_y) for **t=1,000**



Conclusions

- **We presented a fast, efficient and easy to compute numerical method, perfectly suited for multidimensional systems,** in order to check if orbits of symplectic maps and Hamiltonian flows are ordered or chaotic: the computation of the smaller alignment indice (SALI).

- **In 2D maps the SALI tends to zero following completely different time rates**

- **In Hamiltonian flows and in multidimensional maps the SALI tends to zero for chaotic orbits, while in general, it tends to a positive non-zero value for ordered orbits.**

- Usually, when the orbit under consideration is chaotic, the SALI becomes equal to zero, in the sense that it **reaches the limit of the accuracy of the computer**. In that case the chaotic nature of the orbit is established beyond any doubt.

- The use of the SALI helped us decide if an orbit is ordered or chaotic much **faster than the computation of the finite time Lyapunov characteristic number L_t.**